2014

Assessment Task 4
Trial HSC Examination

Mathematics Extension 1

Examiners ~ Mrs D. Crancher, Mr G. Rawson, Mr S. Faulds, Ms P. Biczo

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using a blue or black pen.
- Board approved calculators and mathematical templates and instruments may be used.
- Show all necessary working in Questions 11,12,13 and 14
- This examination booklet consists of 13 pages including a standard integral page and a multiple choice answer sheet.

Total marks (70)

Section I

Total marks (10)

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided on the last page of this question booklet.
- Allow about 15 minutes for this section

Section II

Total marks (60)

- Attempt questions11 to 14
- Answer each question in the writing booklets provided.
- Start a new booklet for each question with your student number (or name) and question number at the top of the page.
- All necessary working should be shown for every question
- Allow about 1 hour 45 minutes for this section

Name and Student Number : ______________________________

Teacher : ______________________________

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Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 – 10.

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1 A curve is defined by the parametric equations $x = \sin 2t$ and $y = \cos 2t$.

Which of the following, in terms of t, equates to $\dfrac{dy}{dx}$?

(A) $\cos 4t$ (B) $2\tan 2t$

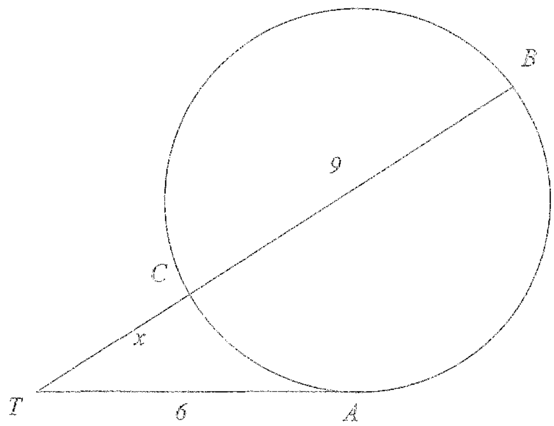
(C) $2\sin 4t$ (D) $-\tan 2t$

2 If two roots of the equation $x^3 - 2x^2 + kx + 18 = 0$ are equal in magnitude but opposite in sign, then what is the value of k ?

(A) -9 (B) -6

(C) 6 (D) 9

3 Line TA is a tangent to the circle at A and TB is a secant meeting the circle at B and C.

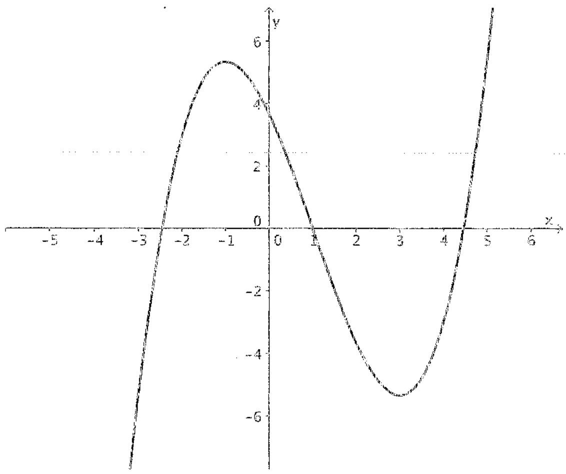


Given that $TA = 6$, $CB = 9$ and $TC = x$, what is the value of x ?

(A) -12 (B) 2

(C) 3 (D) 4

4 The diagram below shows the graph of a function $y = g(x)$.



What is the largest possible domain containing $x = 0$ for which the function will have an inverse function ?

(A) $x \le 1$ (B) $-1 \le x \le 3$

(C) $0 \le x \le 3$ (D) All real x

5 Which expression is a correct expansion of $\sin 4\theta$?

(A) $4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$

(B) $4\sin^3\theta\cos\theta - 4\sin\theta\cos^3\theta$

(C) $4\sin^3\theta\cos^2\theta - 4\sin^2\theta\cos^3\theta$

(D) $4\sin^2\theta\cos^3\theta - 4\sin^3\theta\cos^2\theta$

6 What is the exact value of $\int_0^1 \frac{dx}{1+x^2}$?

(A) $\tan^{-1}\frac{\pi}{4}$ (B) $\ln 2$

(C) $-\frac{1}{2}$ (D) $\frac{\pi}{4}$

7 Which expression is equal to $\int x e^{x^2+5}\,dx$?

(A) $e^{x^2+5} + c$ (B) $\frac{1}{2}e^{x^2+5} + c$

(C) $2e^{x^2+5} + c$ (D) $xe^{x^2+5} + c$

Section II

60 marks
Attempt Questions 11 and 14

Allow about 1 hour 45 minutes for this section

Answer each question in the appropriate writing booklet.

All necessary working should be shown in every question.

Question 11 (15 marks) **Marks**

(a) Solve $|p-2| > \sqrt{2(p-2)}$ **3**

(b) If A and B are the points $(-1,3)$ and $(4,8)$ find the coordinates of the point which divides AB externally in the ratio 3:2. **2**

(c) Show that the acute angle between the two curves $y = x^2$ and $y = x^2 - 2x - 4$ is approximately $4°34'$. **3**

(d) For $n = 1, 2, 3, \ldots$, let $S_n = 1^2 + 2^2 + \ldots + n^2$.

Use mathematical induction to prove that, for $n = 1, 2, 3, \ldots$ **3**

$$S_n = \frac{1}{6}n(n+1)(2n+1)$$

(e) There are three identical blue marbles and four identical yellow marbles arranged in a row.

(i) How many different arrangements are possible ? **1**

(ii) How many different arrangements of just five of these marbles are possible ? **2**

(f) The staff in an office consists of 4 males and 7 females. **1**

How many committees of 5 can be chosen which contain exactly 3 females?

8 Using the substitution $u = 2x+1$, find $\int \frac{1}{(2x+1)^2+1}\,dx$.

(A) $\frac{1}{2}\tan^{-1} x$

(B) $\frac{1}{2}\tan^{-1}(2x+1)$

(C) $\tan^{-1}(2x+1)$

(D) $2\tan^{-1}(2x+1)$

9 What is the derivative of $\log_e\left(\frac{2x}{x-1}\right)$?

(A) $\frac{-2}{2x(x-1)}$

(B) $\frac{x-2}{2x(x-1)}$

(C) $\frac{x-1}{x}$

(D) $\log_e 2$

10 For what values of a is $\frac{a+1}{a} \le 1$?

(A) $a > 0$

(B) $a \ge 0$

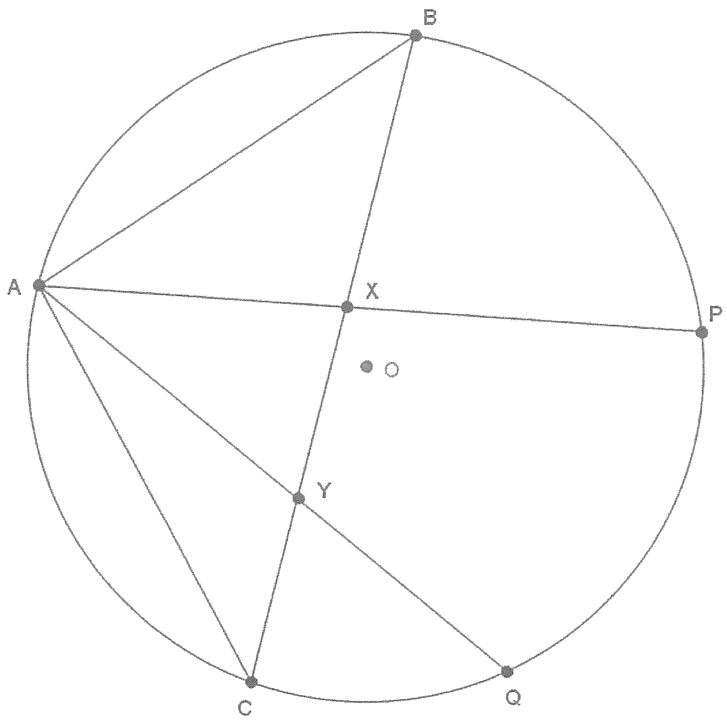
(C) $a < 0$

(D) $-1 \le a < 0$

~ End of Section I ~

(d) Let $ABPQC$ be a circle such that $AB = AC$, AP meets BC at X, and AQ meets BC at Y, as shown below.

Let $\angle BAP = \alpha$ and $\angle ABC = \beta$.



(i) Copy the diagram into your writing booklet, marking the information given above, and state why $\angle AXC = \alpha + \beta$. **1**

(ii) Prove that $\angle BQP = \alpha$ **1**

(iii) Prove that $\angle BQA = \beta$ **1**

(iv) Prove that the quadrilateral $PQYX$ is cyclic **2**

Question 13 (15 marks) **Marks**

(a) Consider the function $f(x)=\dfrac{x}{x-1}$.

(i) Suggest a domain for which the function is continuous and has an inverse. **1**

(ii) Show algebraically that the function is its own inverse. **1**

(b) For the function $y=\dfrac{\pi}{2}+2\sin^{-1}\left(\dfrac{x}{3}\right)$:

(i) State the domain and range. **2**

(ii) Draw a neat sketch of the function showing all important features. **1**

(c) Use trigonometric identities to solve the equation **3**

$\cos 2x-\sin x=0$ for $-\pi\le x\le\pi$.

(d) (i) Express $8\sin x-15\cos x$ in the form $R\cos(x-\phi)$, where ϕ is measured to the nearest degree. **2**

(ii) Hence, or otherwise, solve the equation **2**

$8\sin x-15\cos x=10$ for $0°\le x\le 360°$,

giving your answer/s to the nearest degree.

(e) The derivative of a function is given by $f'(x)=\dfrac{1}{4+9x^2}$ and $f(0)=\dfrac{\pi}{4}$.

Find the value of $f\left(\dfrac{2\sqrt{3}}{9}\right)$, giving your answer in exact form. **3**

Question 14 (15 marks) **Marks**

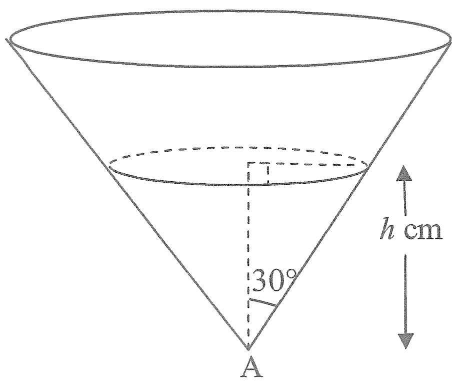
(a) Evaluate $\displaystyle\int_0^{2\ln 3} \frac{e^x}{1+e^x}\,dx$, giving your answer in simplest exact form. **2**

(b) The flow rate of water from a natural spring is given by $\dfrac{dV}{dt} = 0.2e^{-0.04t}$, where V is the volume in megalitres of water, and t the time in days.

(i) At what time is the water flowing at ~~twice~~ half the initial rate? **2**

(ii) How much water will flow from the spring in the first 10 days? **2**

(c)



An egg timer in the shape of an inverted right circular cone with semi vertical Angle $30°$ contains sand to a depth of h cm. The sand flows out of the apex (A) of the cone at a constant rate of 0.5 cm^3/s.

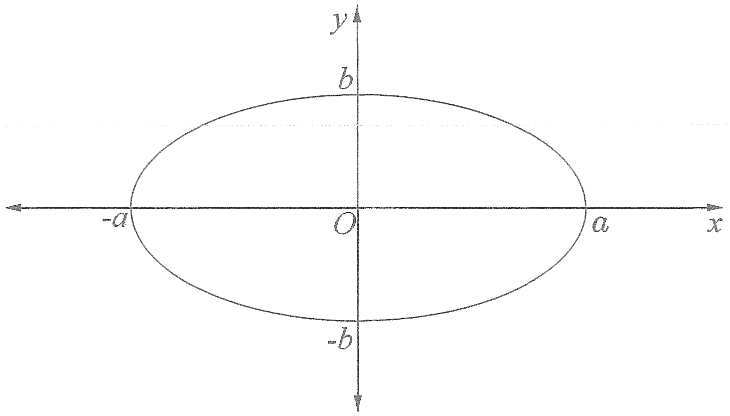
(i) Show that the volume V cm^3 of sand in the cone is given by $V = \dfrac{1}{9}\pi h^3$. **1**

(ii) Find the value of h when the depth of sand in the egg timer is decreasing at a rate of 0.05 cm/s, giving your answer correct to 2 decimal places. **2**

Question 14 continued on nest page...

Question 14 continued...

(f)



The equation of the ellipse shown is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

(i) Show that the area of the ellipse is given by $A=\dfrac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$. **2**

(ii) By using the substitution $x=a\sin\theta$, where $0\le\theta\le\dfrac{\pi}{2}$,

show that the area of the ellipse is πab units^2. **4**

~ End of Section II ~

Year 12 2014	Mathematics Extension 1	Task 4 Trial HSC
Question No. 11	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

HE2 - uses inductive reasoning in the construction of proofs
PE3 - uses problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations

Outcome	Solutions	Marking Guidelines

PE3

11.

a) Solve $|p-2| > \sqrt{2(p-2)}$

Restrictive domain:

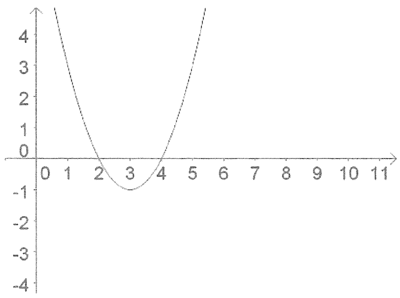
$$2(p-2) \geq 0$$
$$p-2 \geq 0$$
$$p \geq 2$$

RHS only exists for $p \geq 2$.

Now, $|p-2| > \sqrt{2(p-2)}$

Squaring both sides

$$(p-2)^2 > 2(p-2)$$
$$p^2 - 4p + 4 > 2p - 4$$
$$p^2 - 6p + 8 > 0$$
$$(p-2)(p-4) > 0$$



$$p < 2, p > 4$$

Considering the restrictive domain (i.e. $p \geq 2$), the solution is $p > 4$.

Marking Guidelines:

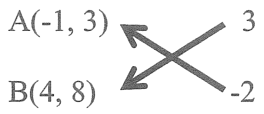
3 marks complete correct solution

2 marks for substantial correct working leading to a correct solution

1 mark for limited correct working leading to a correct solution

b)

Since the division is external find the coordinates of a point dividing AB in the ratio 3:-2



$$x = \frac{(-2)(-1)+(3)(4)}{3+(-2)}, \quad y = \frac{(-2)(3)+(3)(8)}{3+(-2)}$$
$$= 14 \qquad\qquad = 18$$

Thus the coordinates of the point required are (14, 18).

Marking Guidelines:

2 marks complete correct solution

1 mark for substantial correct working leading to a correct solution **or** finding the internal division of the interval AB in the ratio 3:2

PE3	c) $y = x^2$.....................(A) $y = x^2 - 2x - 4$........(B) Sub A into B $x^2 = x^2 - 2x - 4$ $0 = -2x - 4$ $4 = -2x$ when $x = -2,\ y = (-2)^2$ $-2 = x$ $= 4$ The point of intersection is (-2, 4) Gradient of the tangent to $y = x^2$ at $x = -2$ is: $\frac{dy}{dx} = 2x$ $at\ x = -2,\ \frac{dy}{dx} = 2(-2)$ $= -4$ $\therefore m_1 = -4$ Gradient of the tangent to $y = x^2 - 2x - 4$ at $x = -2$ is: $\frac{dy}{dx} = 2x - 2$ $at\ x = -2,\ \frac{dy}{dx} = 2(-2) - 2$ $= -6$ $\therefore m_2 = -6$ Acute angle between the two curves $y = x^2$ and $y = x^2 - 2x - 4$ is: $\tan\theta = \left\lvert \frac{-4-(-6)}{1+(-4)(-6)} \right\rvert$ $= \frac{2}{5}$ $\therefore \theta = 4°34'$	3 marks complete correct solution 2 marks for substantial correct working leading to a correct solution 1 mark for limited correct working leading to a correct solution

HE2

d)
Required to prove that:

$$S_n = \frac{1}{6}n(n+1)(2n+1), \quad \text{for } n = 1,2,3,...$$

Show true for $n = 1$:

$$LHS = (1)^2$$
$$= 1$$
$$RHS = \frac{1}{6}(1)(1+1)(2(1)+1)$$
$$= \frac{6}{6}$$
$$= 1$$
$$LHS = RHS$$

$\therefore$ The statement is true for $n = 1$.

Assume true for $n = k$:

$$S_k = \frac{1}{6}k(k+1)(2k+1)$$

Prove true for $n = k+1$, i.e. prove that:

$$S_{k+1} = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

Now,

$$S_k = S_k + (k+1)^2$$
$$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$
$$= (k+1)\left[\frac{1}{6}k(2k+1) + (k+1)\right]$$
$$= (k+1)\left[\frac{k(2k+1) + 6(k+1)}{6}\right]$$
$$= \frac{1}{6}(k+1)(2k^2 + k + 6k + 6)$$
$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

$\therefore$ Statement is true for $n = k+1$.

If the statement is true for $n = k$, it is also true for $n = k+1$.
Therefore, the statement is true for all natural numbers n
(i.e. $n = 1,2,3,...$).by mathematical induction.

3 marks complete correct solution

2 marks for substantial correct working leading to a correct solution

1 mark for limited correct working leading to a correct solution

PE3	e) (i) Possible number of arrangements (permutations) of 7 objects, 3 of which are identical (3 blue marbles) and the other 4 of which are identical (4 yellow): $$\frac{7!}{3!4!} = 35$$	1 mark for $\frac{7!}{3!4!}$ or 35
PE3	(ii) Possible combinations of 5 marbles are: 3B 2Y 2B 3Y 1B 4Y $$\frac{5!}{3!2!} + \frac{5!}{3!2!} + \frac{5!}{4!} = 25$$	2 marks for complete correct solution 1 marks for substantial correct working that could lead to a correct solution
PE3	f) Each committee will consist of 2 males and 3 females. The number of committees of 5 staff with 3 females will be: $${}^{4}C_{2} \times {}^{7}C_{3} = 210$$	1 mark for ${}^{4}C_{2} \times {}^{7}C_{3}$ or 210

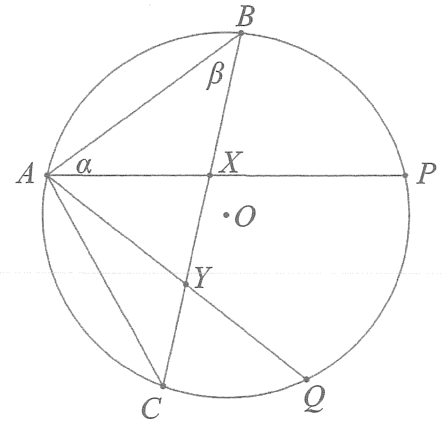
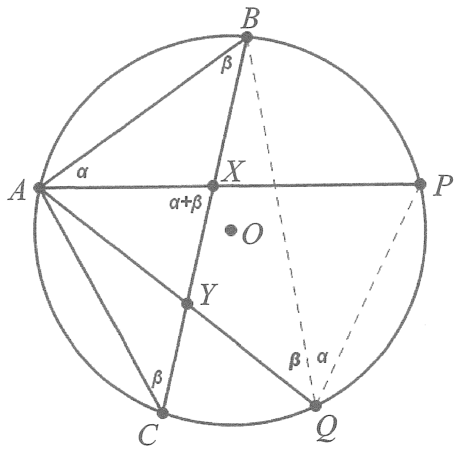
Year 12 Ext 1 Mathematics	TRIAL EXAM 2014
Question No. 12 Solutions and Marking Guidelines	

Outcomes Addressed in this Question

PE3 - solves problems involving polynomials, circle geometry and parametric representations

Outcome	Solutions	Marking Guidelines
PE3	(a) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\gamma}{\alpha\beta\gamma}$ $=\frac{-\frac{2}{5}}{-\left(\frac{-4}{5}\right)}$ $=-\frac{1}{2}$	**2 marks:** correct solution **1 mark:** substantially correct solution
PE3	(b) $f(x)=2x^4+ax^3-2x^2+bx+6$ By the remainder theorem, $f(1)=12$ $2+a-2+b+6=12$ $a+b=6 \quad ...(1)$ By the factor theorem, $f\left(-\frac{1}{2}\right)=0$ $2\left(-\frac{1}{2}\right)^4+a\left(-\frac{1}{2}\right)^3-2\left(-\frac{1}{2}\right)^2+b\left(-\frac{1}{2}\right)+6=0$ $\frac{1}{8}-\frac{a}{8}-\frac{1}{2}-\frac{b}{2}=-6$ $1-a-4-4b=-48$ $a+4b=45 \quad ...(2)$ $a+b=6 \quad ...(1)$ $a+4b=45 \quad ...(2)$ $3b=39 \quad (2)-(1)$ $b=13$ $a=-7$	**3 marks:** correct solution **2 marks:** substantially correct solution **1 mark:** partially correct solution
PE3	(c)(i) tangents are $y=px-ap^2$ and $y=qx-aq^2$ so $px-ap^2=qx-aq^2$ $px-qx=ap^2-aq^2$ $(p-q)x=a(p-q)(p+q)$ $x=a(p+q)$ and $y=p(a(p+q))-ap^2$ $=ap^2+apq-ap^2$ $=apq$ ie tangents meet at $T:(a(p+q),apq)$	**2 marks:** correct solution **1 mark:** substantially correct solution

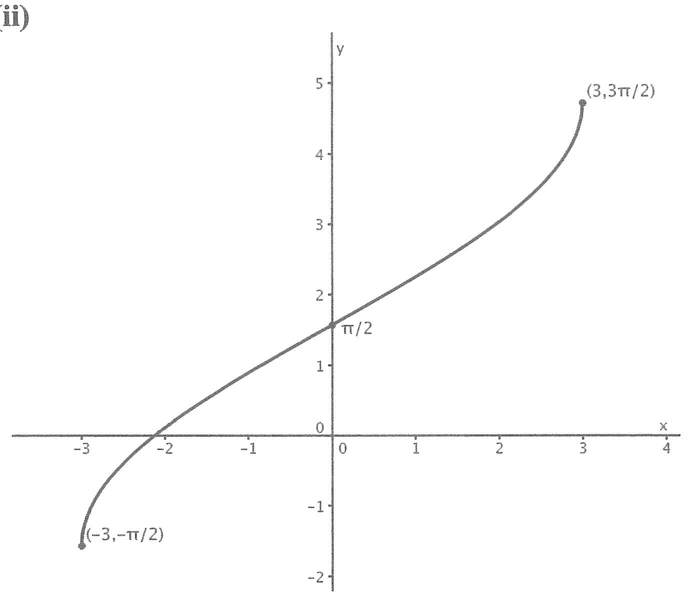
PE3	(c)(ii) gradients of tangents are $m_1 = p$ & $m_2 = q$ $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$ $\tan 45° = \dfrac{p-q}{1+pq}$ $1 = \dfrac{p-q}{1+pq}$ $1 + pq = p - q$	**<u>1 mark:</u>** correct solution
PE3	(c)(iii) $x^2 - 4ay = (a(p+q))^2 - 4a(apq)$ $= a^2(p^2+q^2+2pq) - 4a^2pq$ $= a^2(p^2+q^2-2pq)$ $= a^2(p-q)^2$ $= a^2(1+pq)^2$ [as $p-q = 1+pq$ $= a^2(1+2pq+(pq)^2)$ $= a^2 + 2a(apq) + (apq)^2$ $= a^2 + 2ay + y^2$ {as $y = apq$ locus of T is $x^2 = a^2 + 6ay + y^2$	**<u>2 marks:</u>** correct solution **<u>1 mark:</u>** substantially correct solution NB: as T <u>does not</u> lie on the parabola $x^2 = 4ay$, it does not satisfy the *equation*. Subbing the point into the actual equation is not a valid thing to do, and is not the same as evaluating the *expression* $x^2 - 4ay$.
PE3 (an 'otherwise' method)	(c)(iii) $x = a(p+q)$ ...(1) $y = apq$ ...(2) $p - q = 1 + pq$ $(p-q)^2 = (1+pq)^2$ $p^2 - 2pq + q^2 = 1 + 2pq + (pq)^2$ $p^2 + q^2 = 1 + 4pq + (pq)^2$ ...(3) from (1) $x = a(p+q)$ $x^2 = a^2(p^2+q^2+2pq)$ sub in (3) $= a^2(1+4pq+(pq)^2+2pq)$ $= a^2(1+6pq+(pq)^2)$ sub in (2) $= a^2\left(1+6\left(\frac{y}{a}\right)+\left(\frac{y}{a}\right)^2\right)$ ie, locus of T is $x^2 = a^2 + 6ay + y^2$	

	(d) (i) 	
PE3	$\angle AXC = \angle BAX + \angle ABX$ (exterior angle of ΔBAX equals sum of the two opposite interior angles) $= \alpha + \beta$	**<u>1 mark:</u>** correct solution
PE3	(ii) $\angle BQP = \angle BAP$ (angles at the circumference on the same arc BP are equal) $= \alpha$	**<u>1 mark:</u>** correct solution
PE3	(iii) $\angle BCA = \angle ABC$ (angles opposite equal sides in ΔABC) $\angle BCA = \beta$ $\angle BQA = \angle BCA$ (angles at the circumference on the same arc AB are equal) so $\angle BQA = \beta$	**<u>1 mark:</u>** correct solution
PE3	(iv)  $\angle AXY = \alpha + \beta$ (proved in (i)) $\angle PQY = \angle BQP + \angle BQA$ $= \alpha + \beta$ $= \angle AXY$ $\therefore$ quadrilateral $PQYX$ is cyclic (exterior angle of quadrilateral $PQYX$ equals opposite interior angle)	**<u>2 marks:</u>** correct solution **<u>1 mark:</u>** substantially correct solution

Year 12 Mathematics Extension 1 Task 4 Trial Examination 2014		
Question No. 13	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

HE4 uses the relationship between functions, inverse functions and their derivatives
H5 applies appropriate techniques from the study of calculus, geometry, probability, **trigonometry** and series to solve problems
HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

Outcome	Solutions	Marking Guidelines
HE4	**(a)(i)** Any domain not containing $x = 1$. eg. $x > 1$	**1 mark** Any correct domain, not containing $x = 1$.
HE4	**(ii)** Consider the function as: $y = \frac{x}{x-1}$ Interchanging x and y will give the inverse of the function: $x = \frac{y}{y-1}$ $x(y-1) = y$ $xy - x = y$ $xy - y = x$ $y(x-1) = x$ $y = \frac{x}{x-1}$, which is the original function. ie. the function is its own inverse provided $x \neq 1$.	**1 mark** Correct solution.
HE4	**(b)(i)** Domain: $-1 \leq \frac{x}{3} \leq 1$ $-3 \leq x \leq 3$ Range: $-\frac{\pi}{2} \leq \sin^{-1}\left(\frac{x}{3}\right) \leq \frac{\pi}{2}$ $-\pi \leq 2\sin^{-1}\left(\frac{x}{3}\right) \leq \pi$ $-\frac{\pi}{2} \leq \frac{\pi}{2} + 2\sin^{-1}\left(\frac{x}{3}\right) \leq \frac{3\pi}{2}$ $-\frac{\pi}{2} \leq y \leq \frac{3\pi}{2}$	**2 marks** Correct answers for both domain and range. **1 mark** One of domain or range stated correctly.
HE4	**(ii)**  	**1 mark** Correctly drawn graph showing all important features.

H5, HE7	**(c)** $\cos 2x - \sin x = 0 \qquad -\pi \le x \le \pi$ $\cos^2 x - \sin^2 x - \sin x = 0$ $1 - \sin^2 x - \sin^2 x - \sin x = 0$ $2\sin^2 x + \sin x - 1 = 0$ $2\sin^2 x + 2\sin x - \sin x - 1 = 0$ $2\sin x(\sin x + 1) - (\sin x + 1) = 0$ $(2\sin x - 1)(\sin x + 1) = 0$ $\therefore \sin x = \frac{1}{2}, -1$ $x = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$	**3 marks** Correct solution. **2 marks** Substantial progress towards correct solution. eg. Correctly factorises quadratic in $\sin x$. **1 mark** Some progress towards correct solution. eg. Correct use of identities to form a quadratic equation in $\sin x$.
H5, HE7	**(d)(i)** To obtain an expression in the form $R\cos(x - \phi)$, use the expansion $\cos(A - B) = \cos A\cos B + \sin A\sin B$. $8\sin x - 15\cos x = -15\cos x + 8\sin x$ $= 17\left(\frac{-15}{17}\cos x + \frac{8}{17}\sin x\right)$, since $R = \sqrt{(-15)^2 + 8^2}$ $= 17\cos(x - \phi)$, (where $\cos\phi = \frac{-15}{17}$ and $\sin\phi = \frac{8}{17}$ (an angle in the second quadrant) or $\phi = \tan^{-1}\left(\frac{-8}{15}\right)$ $= 152°$ (to the nearest degree)) $= 17\cos(x - 152°)$ **OR** using the expansion $\cos(A + B) = \cos A\cos B - \sin A\sin B$. $8\sin x - 15\cos x = -(15\cos x - 8\sin x)$ $= -17\left(\frac{15}{17}\cos x - \frac{8}{17}\sin x\right)$, since $R = \sqrt{(-15)^2 + 8^2}$ $= -17\cos(x + \phi)$, (where $\cos\phi = \frac{15}{17}$ and $\sin\phi = \frac{8}{17}$ (an angle in the first quadrant) or $\phi = \tan^{-1}\left(\frac{8}{15}\right)$ $= 28°$ (to the nearest degree)) $= -17\cos(x + 28°)$	**2 marks** Correct solution. **1 mark** Substantial progress towards correct solution.
H5, HE7	**(ii)** $8\sin x - 15\cos x = 10, \ 0° \le x \le 360°$ $17\cos(x - 152°) = 10, \ -152° \le x - 152° \le 208°$ $\cos(x - 152°) = \frac{10}{17}$ $x - 152° = \cos^{-1}\left(\frac{10}{17}\right)$ $= -54°, 54°$ $x = 98°, 206° \qquad 0° \le x \le 360°$	**2 marks** Correct solution from answer obtained in (i) (provided answer in (i) does not make question easier.) **1 mark** Substantial progress towards correct solution.

HE4	**(e)** $f'(x)=\frac{1}{4+9x^2}$ $=\frac{1}{9}.\frac{1}{\frac{4}{9}+x^2}$ $=\frac{1}{9}.\frac{1}{\left(\frac{2}{3}\right)^2+x^2}$ $f(x)=\frac{1}{9}.\frac{3}{2}\tan^{-1}\left(\frac{3x}{2}\right)+c$ $=\frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)+c$ $but\ f(0)=\frac{\pi}{4}$ $\therefore \frac{\pi}{4}=\frac{1}{6}\tan^{-1}(0)+c$ $c=\frac{\pi}{4}$ Now, $f(x)=\frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)+\frac{\pi}{4}$ $f\left(\frac{2\sqrt{3}}{9}\right)=\frac{1}{6}\tan^{-1}\left(\frac{3.2\sqrt{3}}{2.9}\right)+\frac{\pi}{4}$ $=\frac{1}{6}\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)+\frac{\pi}{4}$ $=\frac{1}{6}.\frac{\pi}{6}+\frac{\pi}{4}$ $=\frac{\pi}{36}+\frac{\pi}{4}$ $=\frac{\pi}{36}+\frac{9\pi}{36}$ $=\frac{5\pi}{18}$	**3 marks** Correct solution. **2 marks** Substantial progress towards correct solution, at least correctly stating the correct function, $f(x)$. **1 mark** Some progress towards correct solution, at least demonstrating some knowledge of how to obtain the correct inverse tan primitive.

Year 12 Trial Higher School Certificate	Extension 1 Mathematics	Examination 2014
Question No. 14	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

- H3 Manipulates algebraic expressions involving logarithmic and exponential functions
- H4 Expresses practical problems in mathematical terms based on simple given models
- H5 Applies appropriate techniques from the study of **Calculus**, Geometry, Probability, Trigonometry and Series to solve problems
- H8 Uses techniques of integration to calculate areas and volumes
- H9 Communicates using mathematical language, notation, diagrams and graphs
- HE5 Applies the Chain Rule to problems including those involving velocity and acceleration as functions of displacement
- HE6 Determines integrals by reduction to a standard form through a given substitution
- HE7 Evaluates mathematical solutions to problems and communicates them in an appropriate form

Outcome	Solutions	Marking Guidelines
H3, HE6	(a) $\int \frac{e^x}{1+e^x}\,dx$ is in the form $\int \frac{f'(x)}{f(x)}\,dx$ $\therefore \int_0^{2\ln 3} \frac{e^x}{1+e^x}\,dx = \left[\log\left(1+e^x\right)\right]_0^{2\ln 3}$ $= \log\left(1+e^{2\ln 3}\right) - \log\left(1+e^0\right)$ $= \log\left(1+e^{\ln 3^2}\right) - \log 2$ $= \log(1+9) - \log 2$ $= \log\frac{10}{2} = \ln 5$	2 marks : correct solution 1 mark : significant progress towards answer
H3, H5	(b)(i) $\frac{dV}{dt} = 0.2e^{-0.04t}$ When $t=0$, $\frac{dV}{dt} = 0.2e^0 = 0.2$ $\therefore$ half the initial rate is 0.1. When $\frac{dV}{dt} = 0.1$, $0.1 = 0.2e^{-0.04t}$ $0.5 = e^{-0.04t}$ $-0.04t = \log_e 0.5$ $t = \frac{\log_e 0.5}{-0.04}$ $t = 17.3$ (to 1 decimal place) $\therefore$ flowing at half the initial rate after 17.3 days	2 marks : correct solution 1 mark : significant progress towards answer
H3, H5, H9	(ii) Amount of water flowing from spring in first 10 days $= \int_0^{10} 0.2e^{-0.04t}\,dt$ $= \frac{0.2}{-0.04}\left[e^{-0.04t}\right]_0^{10}$ $= -5\left(e^{-0.4}-1\right)$ $= 1.648$ $\therefore$ 1.648 megalitres has flowed out of the spring in the first 10 days.	2 marks : correct solution 1 mark : significant progress towards answer

H4, H5	(c) (i) Let the radius of the circle with height h be r. Volume sand $= \frac{1}{3}\pi r^2 h$ [1] Using the right triangle, $\tan 30° = \frac{r}{h}$ $\therefore \frac{1}{\sqrt{3}} = \frac{r}{h}$ $\therefore r = \frac{h}{\sqrt{3}}$ Substituting in [1], $V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$ $\therefore V = \frac{1}{3}\pi \times \frac{h^3}{3} = \frac{1}{9}\pi h^3$	1 mark : correct solution
HE5	(ii) $V = \frac{1}{9}\pi h^3$ $\frac{dV}{dh} = \frac{1}{3}\pi h^2$ Using the Chain Rule $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ Given $\frac{dV}{dt} = -0.5$, $-0.5 = \frac{1}{3}\pi h^2 \times \frac{dh}{dt}$ When $\frac{dh}{dt} = -0.05$, $-0.5 = \frac{1}{3}\pi h^2 \times -0.05$ $10 = \frac{1}{3}\pi h^2$ $\frac{30}{\pi} = h^2$ $\therefore h = \sqrt{\frac{30}{\pi}}$ or 3.09 cm/s.	2 marks : correct solution 1 mark : significant progress towards answer
H8	(d) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$ $y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)$ From diagram a, b positive, $\therefore$ equation of curve is $y = \frac{b}{a}\sqrt{a^2 - x^2}$ Area ellipse = 4 times the area in quadrant 1 = 4 $\times$ area between the curve and the x axis, from $x = 0$ to $x = a$. $= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$ $\therefore A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx.$	2 marks: correct solution 1 mark: significant progress towards correct solution

HE6	(ii) $x = a\sin\theta$ $\frac{dx}{d\theta} = a\cos\theta$ When $x = 0,\ 0 = a\sin\theta,\ \theta = 0$ (Given $0 \le \theta \le \frac{\pi}{2}$). When $x = a,\ a = a\sin\theta,\ \theta = \frac{\pi}{2}$ (Given $0 \le \theta \le \frac{\pi}{2}$). As $A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$, $A = \frac{4b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\,.a\cos\theta\,d\theta$ $= \frac{4b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2\cos^2\theta}\,.a\cos\theta\,d\theta$ $= \frac{4b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta.a\cos\theta\,d\theta$ (Given $0 \le \theta \le \frac{\pi}{2}$, *a positive*) $= 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$ Since $\cos 2\theta = 2\cos^2\theta - 1,\ \cos^2\theta = \frac{\cos 2\theta + 1}{2}$ Then $A = 4ab\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{2}\,d\theta$ $= 2ab\int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)\,d\theta$ $= 2ab\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\frac{\pi}{2}}$ $= 2ab\left(\frac{1}{2}\sin\pi + \frac{\pi}{2} - 0\right)$ $= 2ab \times \frac{\pi}{2}$ $= \pi ab$ units^2	4 marks: correct solution 3 marks: substantially correct solution 2 marks : significant progress towards a solution 1 mark : some progress towards simplifying the integral